19.10.18.

PREDAVANJA 5

Ana

HOMOGENA DIF. J-NA SA KONST. KOEFICIJENTIMA

$$\boxed{y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0} \quad (1)$$

$a_1, a_2, \ldots, a_n \in \mathbb{R}$

$G = \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \times \mathbb{R}^n \longrightarrow$ oblast egzistencije i jedinstvenost

→ Kako naći opšte rješenje?

- Treba nam n linearnih rješenja

OJLEROV NAČIN → rješenje tražimo u obliku $y = e^{\lambda x}$

$y' = \lambda \cdot e^{\lambda x}$; $y^{(n)} = \lambda^n \cdot e^{\lambda x}$

→ zamjenjujemo sve u početnu j-nu i dobijamo:

$\lambda^n \cdot e^{\lambda x} + a_1 \lambda^{n-1} e^{\lambda x} + \ldots + a_n e^{\lambda x} = 0$

$e^{\lambda x} (\lambda^n + a_1 \lambda^{n-1} + \ldots + a_n) = 0$

→ uvijek različito od nule

$\Rightarrow$ treba da dobijemo da je $\underbrace{P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n = 0}_{\text{karakteristični polinom početne dif. j-ne}}$

OSNOVNA TEOREMA ALGEBRE:

polinom n-tog stepena ima n nula

→ razlikujemo više slučajeva:

1° SLUČAJ $\lambda_1, \ldots, \lambda_n$ → realni i različiti korijeni karakterističnog polinoma

za $\lambda_1 \rightarrow \varphi_1(x) = e^{\lambda_1 x}$

$\lambda_2 \rightarrow \varphi_2(x) = e^{\lambda_2 x}$

$\vdots$

$\lambda_n \rightarrow \varphi_n(x) = e^{\lambda_n x}$

$\Rightarrow \varphi_1(x), \ldots, \varphi_n(x)$ su rješenja j-ne (1)

→ treba da ispitamo da li su rješenja linearno nezavisna
→ determinanta Vronskog

$$W(\varphi_1, \dots, \varphi_n) = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) & \dots & \varphi_n(x) \\ \varphi_1'(x) & \varphi_2'(x) & \dots & \varphi_n'(x) \\ \vdots & & & \\ \varphi_1^{(n-1)}(x) & \varphi_2^{(n-1)}(x) & \dots & \varphi_n^{(n-1)}(x) \end{vmatrix} =$$

$$= \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \dots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \dots & \lambda_n e^{\lambda_n x} \\ \vdots & & & \\ \lambda_1^{n-1} e^{\lambda_1 x} & \lambda_2^{n-1} e^{\lambda_2 x} & \dots & \lambda_n^{n-1} e^{\lambda_n x} \end{vmatrix} = e^{(\lambda_1+\lambda_2+\dots\lambda_n)x} \cdot \underbrace{\begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix}}_{\text{Van der Mondova determinanta}}$$

$$= e^{(\lambda_1+\lambda_2+\dots+\lambda_n)x} \underbrace{\prod_{n \geq i > j \geq 1} (\lambda_i - \lambda_j)}_{j \neq i \text{ i uvijek } \neq 0} \neq 0$$

→ dobili rješenja → dobili da su linearno nezavisna

$$y = C_1 \varphi_1(x) + \dots + C_n \varphi_n(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \dots + C_n e^{\lambda_n x}$$

$\boxed{2^\circ \text{ SLUČAJ}}$ $\lambda \in \mathbb{R}$ → lambda je korijen višestrukosti $r > 1$

Posmatrajmo:

$$\boxed{y'' + a_1 y' + a_2 y = 0 \quad (2)}$$

→ karakteristični polinom je: $\boxed{k^2 + a_1 k + a_2 = 0}$ → $\boxed{y = e^{\lambda_1 x} \qquad y = e^{\lambda_2 x}}$

– šta ako je $\lambda_1 = \lambda_2$ → ??

$$\frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2} \quad ; \quad \lambda_1 \neq \lambda_2$$ → jeste rješenje jer je linearna kombinacija

$$\frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2} \xrightarrow[\lambda_1 \to \lambda_2]{} \frac{e^{\lambda_2 x}\left(e^{(\lambda_1-\lambda_2)x} - 1\right)}{(\lambda_1 - \lambda_2)\,x} \cdot x = xe^{\lambda_2 x}$$

$$\frac{e^x - 1}{x} \xrightarrow[x \to 0]{} 1$$

→ ako je $\boxed{a)\ \varphi_1(x) = e^{\lambda x}}$ rješenje onda je i $\boxed{b)\ \varphi_2(x) = xe^{\lambda x}}$ rješenje

naslutili da je φ_2 rješenje; a da li je stvarno rješenje?

c) $\varphi_2'(x) = e^{\lambda x} + \lambda x e^{\lambda x}$

d) $\varphi_2''(x) = \lambda e^{\lambda x} + \lambda e^{\lambda x} + \lambda^2 x e^{\lambda x}$

a, b, c i d zamjenjujemo u jednačinu (2)

$$2\lambda e^{\lambda x} + \lambda^2 x e^{\lambda x} + a_1\left(e^{\lambda x} + \lambda x e^{\lambda x}\right) + a_2 x e^{\lambda x} = 0 \,/ \cdot \frac{1}{e^{\lambda x}}$$

$$2\lambda + \lambda^2 + a_1 + \lambda x a_1 + a_2 x = 0$$

$$2\lambda + a_1 + x\left(\lambda^2 + a_1\lambda + a_2\right) = 0$$

$$\underbrace{P'(\lambda)}_{=0} + x\underbrace{P(\lambda)}_{=0} = 0$$

$$P_n(x) = (x-\lambda)^r \cdot P_{n-r}(x)$$

$$\left.\begin{aligned} &P_n(x) = (x-\lambda)^r \cdot P_{n-r}(x) \\ &P_n(\lambda) = 0 \\ &P_n'(\lambda) = \dots = P_n^{(r-1)}(\lambda) = 0 \end{aligned}\right\} \Rightarrow \text{zbog ovoga } P'(\lambda) + xP(\lambda) = 0 \quad T$$

→ pokazali da su $\varphi_1(x)$ i $\varphi_2(x)$ rješenja j-ne (1)

→ da li su φ_1 i φ_2 linearno nezavisna?

→ linearno nezavisna rješenja ako je det. Vronskog $\neq 0$

$$W(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix} = \begin{vmatrix} e^{\lambda x} & xe^{\lambda x} \\ \lambda e^{\lambda x} & (1+\lambda x)e^{\lambda x} \end{vmatrix} = e^{2\lambda x}\begin{vmatrix} 1 & x \\ \lambda & 1+\lambda x \end{vmatrix} =$$

$$\boxed{= e^{2\lambda x} \neq 0}$$

→ rješenja jesu linearno nezavisna

→ opšte rješenje: $y = c_1 \varphi_1(x) + c_2 \varphi_2(x)$

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}, \quad c_1, c_2 \in \mathbb{R}$$

DIFERENCIJALNA JNA N-TOG REDA IMA N KONSTANTI

→ uopštimo ovu priču: kada imamo da je λ korijen višestrukosti r, za taj korijen imamo rješenja:

$$\left.\begin{aligned} \varphi_1(x) &= e^{\lambda x} \\ \varphi_2(x) &= x e^{\lambda x} \\ &\vdots \\ \varphi_r(x) &= x^{r-1} e^{\lambda x} \end{aligned}\right\}$$

rješenja su linearno nezavisna i ona su dio baze prostora

→ za ostalih $n-r$ rješenja ne znamo iz ovog postupka

$\boxed{3^\circ \text{ SLUČAJ}}$

$$\lambda_1 = \alpha + i\beta \to \varphi_1(x) = e^{\lambda_1 x} = e^{(\alpha+i\beta)x}$$
$$\lambda_2 = \alpha - i\beta \to \varphi_2(x) = e^{\lambda_2 x} = e^{(\alpha-i\beta)x}$$

ova rješenja ćemo svesti na realna

$$\lambda_1 = \alpha + i\beta \to \varphi_1(x) = e^{\lambda_1 x} = e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$
$$\lambda_2 = \alpha - i\beta \to \varphi_2(x) = e^{\lambda_2 x} = e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

I $\varphi_1 = \psi_1(x) + i\,\psi_2(x)$; $\psi_1(x) = e^{\alpha x}\cos\beta x$, $\psi_2(x) = e^{\alpha x}\sin\beta x$

II $\varphi_2 = \psi_1(x) - i\,\psi_2(x) = \overline{\varphi_1}(x)$

$$\left.\begin{aligned} \psi_1 &= \frac{1}{2}(\varphi_1(x) + \varphi_2(x)) \\ \psi_2 &= f(\varphi_1(x), \varphi_2(x)) \ldots \end{aligned}\right\}$$

→ ako saberemo I i II rješenja jne (1)

pomnožimo sa i → oduzmemo ...

$$\left.\begin{aligned} \psi_1(x) &= e^{\alpha x}\cos\beta x \\ \psi_2(x) &= e^{\alpha x}\sin\beta x \end{aligned}\right\} \text{ dio fundamentalnog skupa}$$

→ ako su korijeni višestrukosti veće od 1

→ dobijamo kao u slučaju 2

PRIMJERI

1

$$y'' + y' - 2y = 0$$

$$P(\lambda) = \lambda^2 + \lambda - 2 = 0 \longrightarrow \lambda_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$$

$\boxed{\lambda_1 = -2}$; $\boxed{\lambda_2 = 1}$

Realni i različiti $\Rightarrow$ $\varphi_1(x) = e^{-2x}$; $\varphi_2(x) = e^{x}$

$\rightarrow$ našli φ_1 i φ_2 ; ako smo dobro uradili nema potrebe da provjeravamo;

opšte rješenje : $y = C_1 e^{-2x} + C_2 e^{x}$

e^{-2x}, e^{x} $\longrightarrow$ obrazuju fundamentalni skup

2.

$$y^{(5)} - 6y^{(4)} + 9y^{(3)} = 0$$

$\rightarrow$ karak. polinom : $P(\lambda) = \lambda^5 - 6\lambda^4 + 9\lambda^3 = 0$

$$P(\lambda) = \lambda^3(\lambda^2 - 6\lambda + 9) = 0 \longrightarrow \lambda^3(\lambda-3)^2 = 0$$

$\lambda_1 = 0$; višestrukost 3

$\lambda_2 = 3$; višestrukost 2

za $\lambda_1 = 0$ $\longrightarrow$ viš. 3 $\rightarrow$ $\varphi_1(x) = 1$; $\varphi_2(x) = x$; $\varphi_3(x) = x^2$

za $\lambda_2 = 3$ $\longrightarrow$ viš 2 $\rightarrow$ $\varphi_4(x) = e^{3x}$, $\varphi_5(x) = xe^{3x}$

fundamentalni skup

3.

$$y^{(5)} + 8y^{(3)} + 16y' = 0$$

$$P(\lambda) = \lambda^5 + 8\lambda^3 + 16\lambda = 0$$

$$\lambda(\lambda^4 + 8\lambda^2 + 16) = 0$$

$$\lambda(\lambda^2 + 4)^2 = 0$$

$\lambda^2 + 4 = 0$

$\lambda^2 = -4$ $\rightarrow$ $\lambda = \pm 2i$

$\lambda_1 = 0; \ V_1 = 1 \longrightarrow \varphi_1(x) = 1$

$\lambda_2 = 2i; \ V_2 = 2$

$\lambda_3 = -2i; \ V_3 = 2$ $\rightarrow$ moramo naći 4 f je koje će da čine fundamentalni skup

$\varphi_2(x) = e^{0x} \cdot \cos 2x = \cos 2x$

$\varphi_3(x) = e^{0x} \cdot \sin 2x = \sin 2x$

$\varphi_3(x) = x \cdot \cos 2x \ ; \ \varphi_5 = x \sin 2x$

fundamentalni skup

O.R. $y = C_1 + C_2 \cos 2x + C_3 \sin 2x + C_4 x \cos \cancel{2}x + C_5 x \sin 2x$

4. Sastaviti homogenu j-nu što je moguće nižeg reda koja ima rješenja $y_1 = x \sin x$; $y_2 = x e^x \cos x$

a) homogena lin. dif. j-na; b) homogena lin. dif. j-na sa const. koef.

a) najniže 2. reda

ako je y rješenje j-ne y_1 i $y_2 \rightarrow$ linearno zavisni sa y ⇒

$$\begin{vmatrix} y_1 & y_2 & y \\ y_1' & y_2' & y' \\ y_1'' & y_2'' & y'' \end{vmatrix} = 0 \rightarrow \begin{vmatrix} x \sin x & x e^x \cos x & y \\ \sin x + \cancel{x}\cos x & e^x(\cos 2x + x \cos 2x - 2xe^x \cos 2x) & \\ 2\cos x - x \sin x & 1 & \end{vmatrix} = 0$$

$\rightarrow$ dodati ~~sum~~: bi :

$$y'' \cdot \begin{vmatrix} x \sin x & x e^x \cos 2x \\ \sin x + x \cos x & \dots \end{vmatrix} - y' \left| \ \dots \ \right| + y \left| \ \ \right| = 0$$

b) $y_1 = x \sin x \longrightarrow \sin x, \ \cos x, \ x \cos x \longrightarrow V = 2; \ \boxed{\lambda = \pm i}$

$y_2 = x e^x \cos x \longrightarrow e^x \cos x$; ~~$V = 2$~~,

za $\alpha + i\beta \longrightarrow e^{\alpha x} \cos \beta x$

$\longrightarrow e^{\alpha x} \sin \beta x$

$\beta = 1 \ ; \ \alpha = 0$;

$y_2 = x e^x \cos \cancel{2}x \rightarrow V = 2$; $\alpha = 1; \beta = 1$

$\boxed{\lambda = 1 \pm i}$ $\longrightarrow$

$$P(\lambda) = \underbrace{(\lambda^2+1)^2}_{\text{odgovara } \lambda=\pm i} \left((\lambda-1)^2+1\right)^2 \rightarrow P_1(\lambda) = (\lambda-(1+i))(\lambda-(1-i)) =$$

$$= (\lambda-1-i)(\lambda-1+i) =$$

$$= (\lambda-1)^2 - \underbrace{i^2}_{+1}$$

$$P(\lambda) = (\lambda^4+2\lambda^2+1)(\lambda^2-2\lambda+2)^2 =$$

$$= (\lambda^4+2\lambda^2+1)(\lambda^4+4\lambda^2+4+4\lambda^2-4\lambda^3-8\lambda) = 0$$

Zadatak za vježbu

pod a) naći da je prvo rješ.

pod b) naći da je drugo rješ.

OJLEROVA DIFERENCIJALNA J-NA

$$(Ax+B)^n y^{(n)} + a_1(Ax+B)^{n-1} y^{(n-1)} + \dots + a_n y = 0 \qquad A \neq 0$$

$$\longrightarrow \tilde{x} = Ax + B$$

$$y = y(x); \quad y' = \frac{dy}{dx} = \frac{dy}{d\tilde{x}} \cdot \frac{d\tilde{x}}{dx} \rightarrow A$$

$$y' = A \cdot y'_{\tilde{x}}$$

$$y'' = \frac{dy'}{dx} = A \cdot y''_{\tilde{x}} \cdot A = A^2 \cdot y''_{\tilde{x}}$$

$$\vdots$$

$$y^{(n)} = A^n \cdot y^{(n)}_{\tilde{x}}$$

$\longrightarrow$ kad zamijenimo sve u početnu:

$$\Rightarrow \tilde{x}^n A^n y^{(n)}_{\tilde{x}} + a_1 \tilde{x}^{n-1} A^{n-1} y^{(n-1)}_{\tilde{x}} + \dots + a_n y = 0 \; / : A^n$$

$$\longrightarrow \tilde{x}^n y^{(n)}_{\tilde{x}} + \frac{a_1}{A} \tilde{x}^{(n-1)} \cdot y^{(n-1)}_{\tilde{x}} + \dots + \frac{a_n}{A^n} \cdot y = 0$$

$$\underbrace{x^n y^{(n)} + p_1 x^{n-1} \cdot y^{(n-1)} + \dots + p_n y = 0}_{\text{ovo smo izvodili jer nam je lakše tako}} \qquad G_1 = (-\infty, 0) \quad G_2 = (0, +\infty)$$

Smjena: $\underbrace{x=e^{z}}_{\text{1. slučaj}}$; $G_1 \dot= (-\infty, 0)$; $G_2 = (0, +\infty)$

$z = \ln x$ $\quad x>0 \quad G_2$

$z = +\ln(-x)$ $\quad x<0 \quad G_1$ → 2. slučaj

→ ovaj slučaj nećemo raditi

1. slučaj

$$x = e^{z}$$

$$y = y(x) \longrightarrow x = e^{z} \longrightarrow y = y(z)$$

$$y' = \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = e^{-z}\cdot y'_z$$

$$y'' = \frac{dy'}{dz}\cdot\frac{dz}{dx} = \frac{d(e^{-z}\cdot y'_z)}{dz}\cdot e^{-z} = (-e^{-z}y'_z + e^{-z}y''_z\, e^{-z}) =$$

$$= e^{-2z}(y''_z - y'_z)$$

$$\vdots$$

$$y^{(n)} = e^{-n\cdot z}\cdot f(y'_z, y''_z, \dots, y^{(n)}_z)$$

→ zamijenimo sve ovo što smo dobili u početnu j-nu

$$\underbrace{\cancel{e^{nz}}\cdot\cancel{e^{-nz}}\cdot f_1(y'_z, \dots, y^{(n)}_z) + p_1 \cancel{e^{(n-1)z}}\cdot\cancel{e^{-(n-1)}}z\cdot f_2(y'_z, \dots, y^{(n-1)}_z) + \dots + p_n\cdot y = 0}$$

→ dif. j-na sa konstantnim koef. po z;

rješenje $\boxed{e^{\lambda z}}$

$$\lambda\in\mathbb{R} \Rightarrow e^{\lambda z} = (\underbrace{e^{z}}_{x})^{\lambda} = x^{\lambda} \longrightarrow$$

vratiti smjenu !

II $z = \ln x \Rightarrow dz = \frac{1}{x} dx$

$$y' = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \cdot y'_z$$

$$y'' = \frac{dy'}{dx} = -\frac{1}{x^2} y'_z + \frac{1}{x} \cdot \frac{dy'_z}{dz} \cdot \frac{dz}{dx} = \frac{1}{x^2}\left(y''_z - y'_z\right)$$

→ slično kao ovamo; vratimo; skrati se → dobijemo potpuno istu dif. j-nu

→ postoji i **III način** za rješ. dif. j-ne u obliku x^λ

$$y = x^\lambda \to y' = \lambda x^{\lambda-1} \to y'' = \lambda(\lambda-1)\, x^{\lambda-2} \ldots$$

$$\ldots\; y^{(n)} = \lambda(\lambda-1)\ldots(\lambda-(n-1)) \cdot x^{\lambda-n}$$

→ kad zamijenimo:

$$x^n \cdot \lambda(\lambda-1)\ldots(\lambda-(n-1))x^{\lambda-n} + p_1 x^{n-1} \cdot (\lambda)(\lambda-1)\ldots(\lambda-(n-1))x^{\lambda-(n-1)} -$$

$$+ \ldots + P_n x^\lambda = 0$$

$$\boxed{x^\lambda \, P_n(\lambda) = 0}$$

$x = 0$ → singularno rješenje jer ne pripada ovoj oblasti egzistencije

$P_n(\lambda)$ → novi karakteristični polinom;

1° $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ → različiti

Rješenja
$$\begin{cases} \varphi_1(x) = x^{\lambda_1} \\ \varphi_2(x) = x^{\lambda_2} \\ \vdots \\ \varphi_n(x) = x^{\lambda_n} \end{cases}$$
→ opšte rj. linearna kombinacija

fundamentalni skup;

2° λ, višestrukosti r:

Rješenja $x^\lambda, \; x^\lambda \ln x, \; x^\lambda \ln^2 x, \ldots, x^\lambda \ln^{r-1} x$

$x = e^z$

$\lambda, R \longrightarrow e^{\lambda z}, \; z e^{\lambda z}, \; z^{r-1} e^{\lambda z}$

$\longrightarrow x^{\lambda}, \; \ln x \cdot x^{\lambda}, \; x^{\lambda}(\ln x)^2, \ldots, x^{\lambda} \ln^{r-1} x$

3° $\lambda = \alpha \pm i\beta$

$e^{\alpha z} \cdot \cos \beta z \longrightarrow x^{\alpha} \cos(\beta \ln x)$

$e^{\alpha z} \cdot \sin \beta z \longrightarrow x^{\alpha} \sin(\beta \ln x)$ $\longrightarrow$ rješenja

Primjeri

$x^2 y'' - 4xy + 6y = 0$

$x^3 y''' + xy' - y = 0$

$(2x+3)^3 y''' + 3(2x+3)y' - 6y = 0$